§ 16.1-2 Line Integrals

Introduction

- A Line Integral is an integral defined on a curve...

There are 4 equivalent ways to define a line integral, and they all mean the same thing:

Theorem: The following four are equivalent –

$$\int_{C} \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}\cdot\vec{v}\,dt = \int_{C} \vec{F}\cdot d\vec{r} = \int_{C} M\,dx + N\,dy + P\,dz$$

We make sense of each of these four expressions for Line Integral

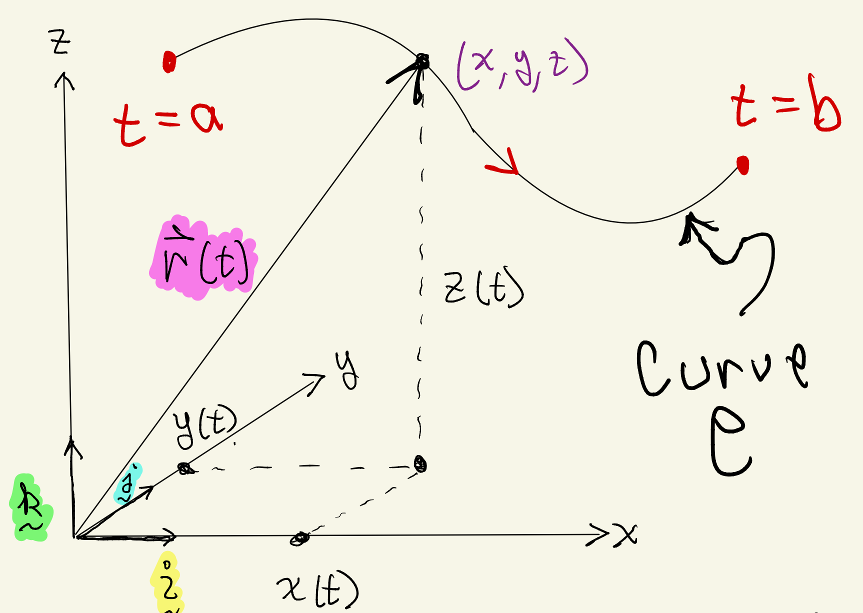
- The first expression gives the meaning of a line integral —

$\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds$ In physics this is called "The work done by the force $\vec{F}$ along the curve $\mathcal{C}$"

Mathematically it is the "Total amout of $\vec{F}$ pointing tangent to the curve $\mathcal{C}$"

- To define $\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds$, recall how we describe a curve $\mathcal{C}$ with orientation

$\mathcal{C}$: $\vec{r} = \vec{r}(t)$
$a \leq t \leq b$



To describe a curve you must give a parameterization

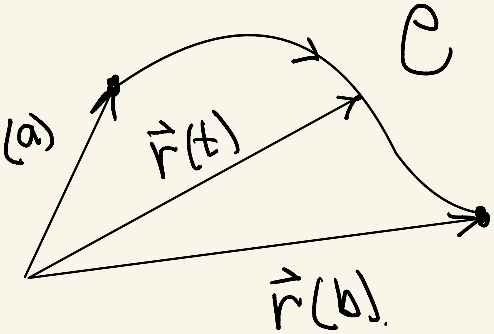
$\vec{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j} + z(t)\,\underset{\sim}{k}$ (oriented correctly)

A curve $\mathcal{C}$ is given by a parameterization

$$\vec{r}(t) = \overrightarrow{(x(t), y(t), z(t))}, \quad a \le t \le b$$

Notation: $\underset{\sim}{x} = \overrightarrow{(x, y, z)}$

There are many ways to parameterize the same curve $\mathcal{C}$



I.e., given $\vec{r}(t)$, $a \le t \le b$,

if $t = \varphi(u)$,

then $\vec{r}(\varphi(u)) = (\vec{r} \circ \varphi)(u)$ $\quad u_a \le u \le u_b$

$b = \varphi(u_b)$

is another parameterization

$a = \varphi(u_a)$

As long as $\varphi'(u) > 0$, one parameterization is as good as another.

- Mathematicians think of different parameterizations of a curve as different coordinate systems on the same curve $\mathcal{C}$

I.e., they give you a way to "name" the points on $\mathcal{C}$ by number t: $P = \vec{r}(t)$

point on curve $\quad$ name

- There is <u>one</u> special parameterization determined by the curve: Namely, arclength parameterization

$$s = \int_a^t \|\vec{v}(\xi)\| d\xi = \varphi(t)$$

Problem: You typically need to start with a parameterization $\vec{r}(t)$ to recover $\varphi(t)$ and thereby obtain the arclength parameterization

$$\vec{r}(s) \equiv \vec{r}(\varphi^{-1}(s)) \quad 0 \le s \le \varphi^{-1}(b)$$

- Important Point: The line integral is independent of parameterization in the sense that it can be computed in different coordinate systems (parameterizations) but you always get the same answer!

- End Introduction to Line Integrals

We begin by defining the line integral in terms of the arclength parameterization –

Given – A vector field $\vec{F}$ & curve C

$$\vec{F}(x,y,z) = M(x,y,z)\,\underset{\sim}{i} + N(x,y,z)\,\underset{\sim}{j} + P(x,y,z)\,\underset{\sim}{k}$$

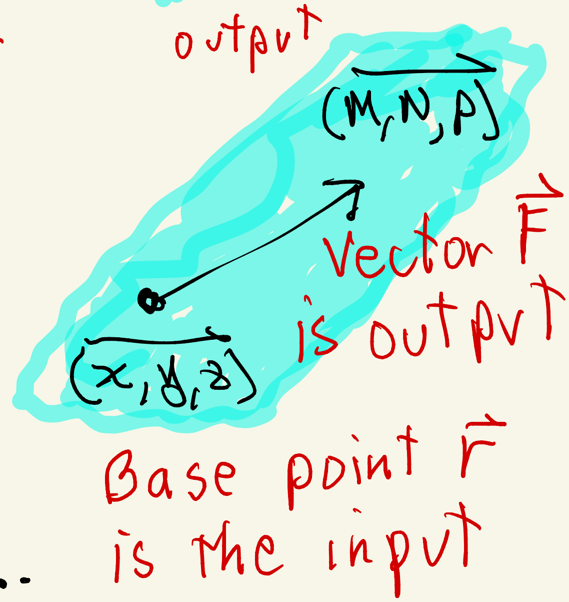
$$= \overrightarrow{(M,N,P)}$$

(Think of $\vec{F}$ as a force field)

Mathematically: $\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\underset{\text{input}}{\overrightarrow{(x,y,z)}} \longmapsto \underset{\text{output}}{\overrightarrow{(M,N,P)}}$$

"$\vec{F}$ assigns a vector $\overrightarrow{(M,N,P)}$ to each point $(x,y,z) \in \mathbb{R}^3$." To be consistent, we treat inputs & outputs as vectors...

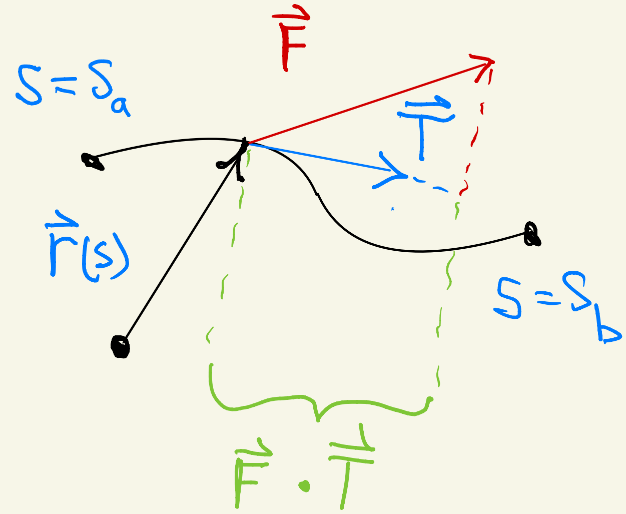


so treat (x,y,z) as a vector $\overrightarrow{(x,y,z)}$

Steps to defining the Line Integral $\int_C \vec{F}\cdot\vec{T}\,ds$

(1) Use the arclength parameterization

- $\vec{F} = \vec{F}(\vec{r}(s))$ is the "force" at $\vec{r}(s)$
- $\vec{T} = \vec{T}(\vec{r}(s))$ is the unit tangent at $\vec{r}(s)$



- $\vec{F}\cdot\vec{T} = \vec{F}(\vec{r}(s))\cdot\vec{T}(\vec{r}(s))$ is the length of the component of $\vec{F}$ in direction $\vec{T}$

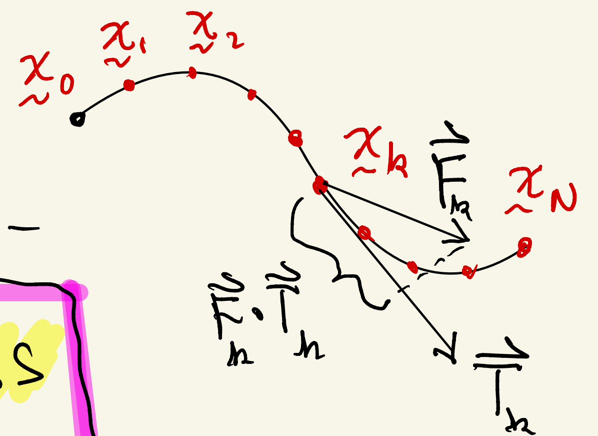
(2) Discretize to define a Riemann Sum

$$s_0 = s_a < s_1 < s_2 < \cdots < s_N = s_b, \quad \Delta s = \frac{s_b - s_a}{N}$$

$$s_k = s_a + k\Delta s, \quad \underset{\sim}{x}_k = \vec{r}(s_k)$$

$$\vec{F}_k = \vec{F}(\underset{\sim}{x}_k)$$

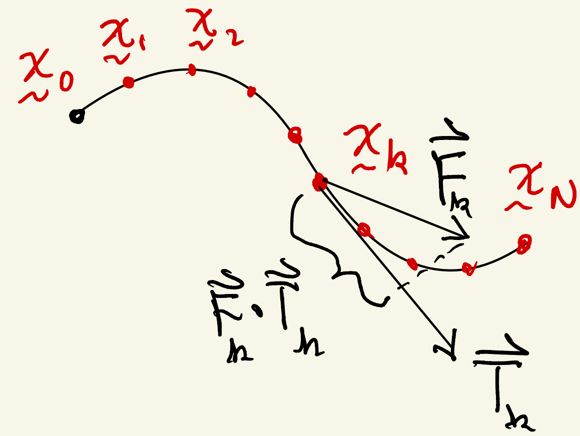
(3) Define Integral as the limit of Riemann Sum –



$$\boxed{\int_C \vec{F}\cdot\vec{T}\,ds = \lim_{N\to\infty} \sum_{k=1}^{N} \vec{F}_k\cdot\vec{T}_k\,\Delta s}$$

$$\vec{T}_k = \vec{T}(\underset{\sim}{x}_k)$$

Defn: $\int_{C} \vec{F}\cdot\vec{T}\,ds = \lim_{N\to\infty} \sum_{k=1}^{N} \vec{F}_k\cdot\vec{T}_k\,\Delta s$

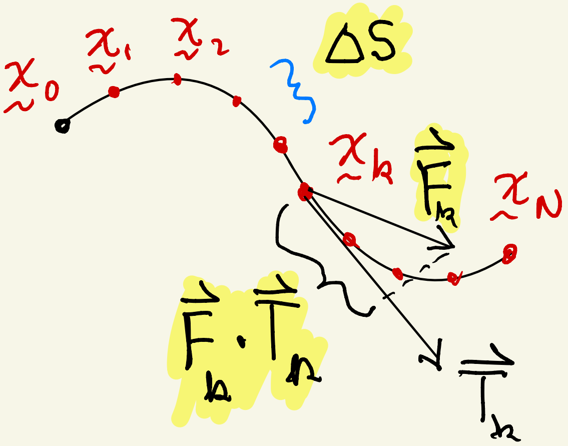


- This gives the simplest most direct meaning of the Line Integral as "The total amount of $\vec{F}$ pointing tangent to C"
- In Physics this is the "sum of the component of force in direction of displacement times displacement, summed along C in a limiting sense"

I.e., the "Total Work Done by $\vec{F}$ along C"

- __Note__: Since arclength parameter is unique, $\int_C \vec{F}\cdot ds$ depends __only__ on force $\vec{F}$ & Curve C.

$$\int_C \vec{F}\cdot\vec{T}\,ds = \lim_{N\to\infty}\sum_{k=1}^{N} \underbrace{\vec{F}_k\cdot\vec{T}_k}_{\text{Component of the Force in direction of displacement}}\ \underbrace{\Delta s}_{\text{Displacement}}$$



Conclude: In Physics, Work is Force × Displacement. When the force is changing along a variable curve, we break the work up into approximate constant force $\vec{F}_k\cdot\vec{T}_k$ times displacement Δs & sum

$\Rightarrow$ Work Done is a Line Integral

- Problem: How do you compute the line integral?

Answer: Use a parameterization!

$$\underbrace{\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds}_{\text{Gives the meaning of line integral as the work done}} = \underbrace{\int_a^b \vec{F}(\vec{r}(t))\cdot\vec{v}(t)\,dt}_{\text{Tells how to compute the line integral - a Math 21B integral}}$$

Important - Each parameterization gives you a different Math 21B integral, but the answer is the same number - namely "The work done by $\vec{F}$ along $\mathcal{C}$"

- How it works: Assume an oriented curve (10)

C is given by parameterization...

$\vec{r}(t) = x(t)\,\underset{\sim}{i} + y(t)\,\underset{\sim}{j} + z(t)\,\underset{\sim}{k} \qquad a \le t \le b$

(1) Discretize $[a, b]$

$t_0 = a < t_1 < t_2 < \cdots < t_k < \cdots t_N = b, \quad \Delta t = \frac{b-a}{N}$

(2) Convert to arc length $ds = \|\vec{v}(t)\|\,dt$

So $\quad \Delta s_k \approx \|\vec{v}(t_k)\|\,\Delta t$

(3) Construct Riemann Sum for Line Integral

$$\int_C \vec{F}\cdot\vec{T}\,ds = \lim_{N\to\infty} \sum_{k=1}^{N} \vec{F}_k\cdot\vec{T}_k\,\Delta s_k$$

(4) Write as a Riemann Sum in t:

$\vec{F}_k = \vec{F}(\vec{r}(t_k)), \quad \vec{T}_k = \frac{\vec{v}(t_k)}{\|\vec{v}(t_k)\|}, \quad \Delta s = \|\vec{v}(t_k)\|\Delta t$

Riemann Sum in t!

$$\sum_{k=1}^{N} \vec{F}_k\cdot\vec{T}_k\,\Delta s_k = \sum_{k=1}^{N} \vec{F}(\vec{r}(t_k))\cdot\frac{\vec{v}(t_k)}{\cancel{\|\vec{v}(t_k)\|}}\cdot\cancel{\|v(t_k)\|}\,\Delta t$$

(5) $$\int_C \vec{F}\cdot\vec{T}\,ds = \lim_{N\to\infty}\sum_{k=1}^{N} \vec{F}(\vec{r}(t_k))\cdot\vec{v}(t_k)\,\Delta t = \int_a^b \vec{F}\cdot\vec{v}\,dt$$

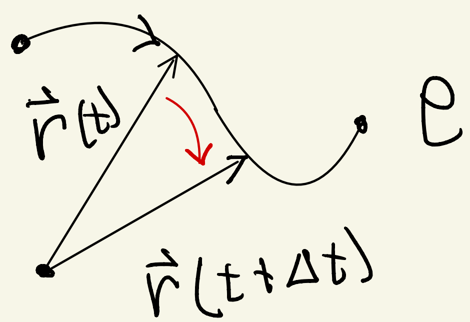
A Math 21B Integral!

(Same answer for any parameterization!)

Conclude:

$$\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{v}(t)\,dt$$

- Holds for any parameterization which respects the orientation of $\mathcal{C}$ (I.e., $\vec{r}(t)$ moves forward on $\mathcal{C}$ as t increases)



- $\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds$ gives the meaning
- $\int_a^b \vec{F}\cdot\vec{v}\,dt$ tells how to compute it
- Since $\int_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds$ is defined in terms of arclength, it has a single value independent of parameterization –

Conclude: Every parameterization gives the same answer!

Example ①

Use Liebniz theory of differentials to "prove" that

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{v}\,dt = \int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz$$

Soln:

$\dfrac{ds}{dt} = \|\vec{v}\|$ so $ds = \|\vec{v}\|\,dt$

$\vec{v} = \dfrac{ds}{dt}\vec{T}$ so $\vec{T}\,ds = \vec{v}\,dt$

$\vec{v} = \dfrac{d\vec{r}}{dt}$ so $d\vec{r} = \vec{v}\,dt$

$\dfrac{d\vec{r}}{dt} = \overrightarrow{\left(\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right)}$ so $d\vec{r} = \overrightarrow{(dx, dy, dz)}$

Thus:

$$\int_C \vec{F}\cdot\underbrace{\vec{T}\,ds}_{\vec{T}\,ds = \vec{v}\,dt} = \int_C \vec{F}\cdot\underbrace{\vec{v}\,dt}_{d\vec{r}} = \int_C \vec{F}\cdot d\vec{r}$$

$\overrightarrow{(M, N, P)} \cdot \overrightarrow{(dx, dy, dz)}$

$$= \int_C M\,dx + N\,dy + P\,dz$$

Conclude: The four ways of writing the line integral are all equivalent:

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{v}\,dt = \int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz$$

When computing, all roads lead to the same answer!

Example ② Let $\mathcal{C}$ be the parabola 14

$y = x^2$, $0 \leq x \leq 1$. Let $\vec{F} = y^2 \underset{\sim}{i} + x \underset{\sim}{j}$.

Evaluate $\displaystyle\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds$

Solution: (1) first step is to find a parameterization of $\mathcal{C}$

Set $t = x$. Then $\vec{r}(t) = \overrightarrow{(x(t), y(t))} = \overrightarrow{(t, t^2)}$

So $\vec{F}(\vec{r}(t)) = \overrightarrow{(t^4, t)}$, $\vec{v}(t) = \overrightarrow{(1, 2t)}$

(2) Use Liebniz differential identities to convert line integral to a Math 21B integral

$$\int_{\mathcal{C}} \vec{F} \cdot \underbrace{\vec{T}\, ds}_{\vec{v}\, dt} = \int_{a=0}^{b=1} \vec{F} \cdot \vec{v}\, dt = \int_0^1 \overrightarrow{(y(t)^2, x(t))} \cdot \overrightarrow{(1, 2t)}\, dt$$

$$= \int_0^1 \overrightarrow{(t^4, t)} \cdot \overrightarrow{(1, 2t)}\, dt = \int_0^1 t^4 + 2t^2\, dt$$

$$= \frac{t^5}{5} + \frac{2t^3}{3}\Bigg]_0^1 = \frac{1}{5} + \frac{2}{3} = \frac{13}{15}$$

Note: we could just as well use

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} \quad \text{or} \quad \int_C \vec{F}\cdot\vec{T}\,ds = \int_C M\,dx + N\,dy + P\,dz$$

to compute - they lead to <u>same</u> t-integral

<u>Example</u>: We had $\vec{F} = \overrightarrow{(y^2, x)}$, $\vec{r}(t) = \overrightarrow{(t, t^2)}$
$0 \le t \le 1$

So $\int_C \vec{F}\cdot d\vec{r} = \int_C \overrightarrow{(M, N)}\cdot\overrightarrow{(dx, dy)}$

$d\vec{r} = \overrightarrow{(dx, dy, dz)}$ $\quad \vec{F} = \overrightarrow{(M, N)} = \overrightarrow{(y^2, x)}$

$$= \int_C M\,dx + N\,dy = \int_C y^2\,dx + x\,dy$$

But $x = t$ so $dx = dt$, $y = t^2$ so $dy = 2t\,dt$

$$\Rightarrow \quad = \int_C t^4\,dt + t\cdot 2t\,dt$$

$$= \int_0^1 t^4 + 2t^2\,dt = \cdots = \frac{13}{15}$$

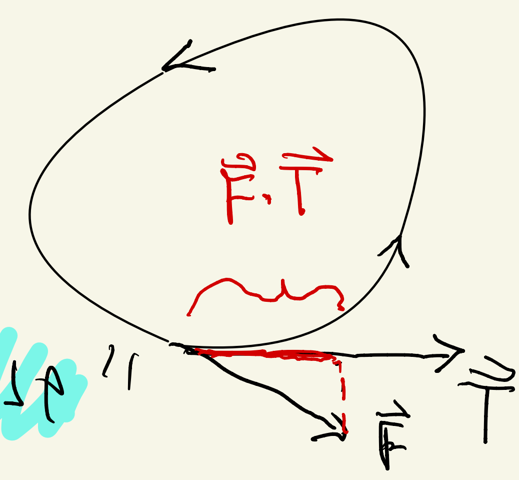
same integral!

Example ③ A simple closed curve is a curve $\vec{r}(t)$, $a \le t \le b$ which is closed ($\vec{r}(a) = \vec{r}(b)$) and simple means it does not cross itself.

Eg. Circle $\vec{r}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$ is a simple closed curve (SCC)

Defn: The line integral of $\vec{F}$ around a closed curve C is called the circulation in $\vec{F}$ around C

I.e., $\oint_C \vec{F} \cdot \vec{T}\, ds$ measures "the total amount of $\vec{F}$ pointing counterclockwise"



Example 3 (cont) Let $\mathcal{C}$ be the circle of radius 2, center $(0,0)$, oriented counter-clockwise. Let $\vec{F} = (x-y)\,\underset{\sim}{i} + x\,\underset{\sim}{j}$. Find the circulation in $\vec{F}$ around $\mathcal{C}$.

Solution: 1 Get a parameterization:

So $\vec{r}(t) = 2\overrightarrow{(\cos t, \sin t)}$, $0 \leq t \leq 2\pi$

2 Circulation $= \int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds$

3 Use Liebniz differentials to set up Math210 integral:

$$\int_{\mathcal{C}} \vec{F} \cdot \vec{T}\, ds = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{v}(\vec{r}(t))\, dt$$

$\vec{F}(\vec{r}(t)) = \overrightarrow{(x(t)-y(t), x(t))} = 2\overrightarrow{(\cos t - \sin t, \cos t)}$

$\vec{v}(t) = \vec{r}\,'(t) = 2\overrightarrow{(-\sin t, \cos t)}$

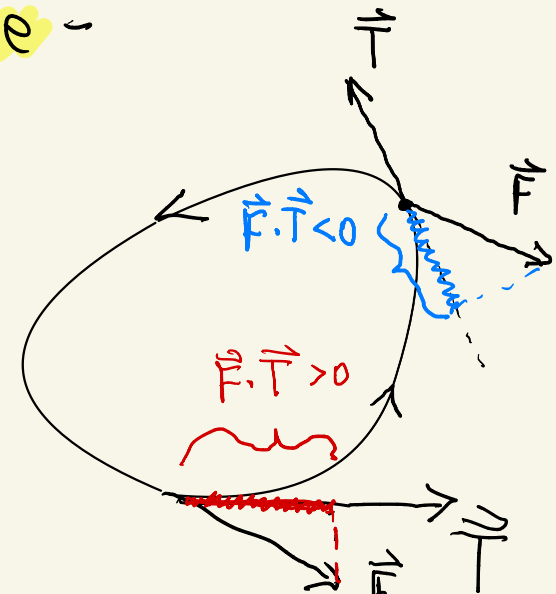
$\vec{F} \cdot \vec{v} = 4\overrightarrow{(\cos t - \sin t, \cos t)} \cdot (-\sin t, \cos t) = 4(-\cos t \sin t + \underbrace{1}_{})$

$\vec{F} \cdot \vec{v} = 4\overrightarrow{(\cos t - \sin t, \cos t)} \cdot (-\sin t, \cos t) = 4(-\cos t\, \overbrace{\sin t + 1)}^{1}$

4 $\int_0^{2\pi} \vec{F} \cdot \vec{v}\, dt = 4\int_0^{2\pi} 1 - \cos t \sin t\, dt = 4\left(t + \frac{\cos^2 t}{2}\right)\Big]_0^{2\pi} = \boxed{8\pi}$

Q: If $\vec{F}$ were the force on a frictionless bead confined to a wire circle in Example ③, which way would the bead circulate?

Ans: If $\oint_e \vec{F}\cdot\vec{T}\,ds > 0$, the "net force" on bead is counterclockwise — if negative, the net force is clockwise.



Since we calculated

$$\int_e \vec{F}\cdot\vec{T}\,ds = 8\pi > 0,$$

the bead would rotate counterclockwise!